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Class groups of quadratic fields of 3-rank at least 2: Effective bounds

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Abstract

In this paper, we give parametric families of both real and complex quadratic number fields whose class group has 3-rank at least 2. As a consequence, we obtain that for all large positive real numbers x, the number of both real and complex quadratic fields whose class group has 3-rank at least 2 and absolute value of the discriminant $\leqslant x$ is $> cx^{1/3}$, where c is some positive constant.

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1. Introduction

Let d be a positive square-free integer. Let $\mathbb{K}$ be either one of $\mathbb{Q}(\sqrt{d})$ or $\mathbb{Q}(\sqrt{-d})$ and let $\Delta_{\mathbb{K}}$ be its discriminant. Given a positive integer g, Murty [16] looked at the issue of constructing "many" such fields whose class number $h_{\mathbb{K}}$ is a multiple of g. In [16], he showed that if g is odd then given a large positive real number x the number of real $\mathbb{K}$ such that $\Delta_{\mathbb{K}} \leqslant x$ and $g \mid h_{\mathbb{K}}$ is $\gg x^{1/(2g)-\varepsilon}$. This was improved to $\gg x^{1/g-\varepsilon}$ by Yu [20]. For the particular case $g = 3$, this was improved to $\gg x^{5/6}$ by Chakraborty and Murty [7] and later to $\gg x^{7/8}$ by Byeon and Koh [3]. In the same paper, Murty also obtained similar lower bounds (with different exponents)

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for the case of the complex quadratic fields with $|\Delta_{\mathbb{K}}| \leqslant x$, which were subsequently improved by Soundararajan [17]. General results of this type for algebraic number fields of an arbitrary degree can be found in [1].

Given a positive integer n and a finite abelian group G we say that G has *n-rank r* if $(\mathbb{Z}/n\mathbb{Z})^r \subseteq G$ and r is the largest positive integer with the property that the previous containment holds. Clearly, if G has an element of order n, then G has n-rank at least 1. The purpose of this paper is two-fold.

One of its goals is to carry out the program initiated by Murty a step further and prove the following result.

Theorem 1. *There exists a positive constant c such that if $x > x_0$ then there are $\geqslant cx^{1/3}$ real quadratic number fields $\mathbb{K}$ with $\Delta_{\mathbb{K}} \leqslant x$ whose class group has 3-rank at least 2. The same result is true for complex quadratic number fields with $|\Delta_{\mathbb{K}}| \leqslant x$.*

We point out that the same result as our Theorem 1 above was obtained recently by Byeon [4] for the case of imaginary quadratic fields. Namely, Byeon showed that for all odd integers $g \geqslant 3$, there are at least $x^{1/g-\varepsilon}$ imaginary quadratic fields of discriminant $< x$ in absolute value whose g rank is at least 2 once $x > x_0(\varepsilon)$. Here, $\varepsilon > 0$ can be arbitrarily small. Thus, his bound is $x^{1/3-\epsilon}$ once $x > x_0(\varepsilon)$ for any $\varepsilon > 0$ for the 3-rank at least two case. Presumably, Byeon's method can be strengthened to remove the dependence on $\varepsilon > 0$ by combining our arguments (based on Theorem 5) with his construction.

We recall that Craig [5] (see also [6]) modified a construction of Yamamoto [19] and gave a parametric family of complex quadratic fields whose class group has g-rank at least two. Using his result to count the number of distinct imaginary quadratic fields of discriminant $\leqslant x$ in absolute value with class group having 3-rank at least two, one needs to count the number of distinct imaginary quadratic fields of the form $\mathbb{Q}(\sqrt{d})$ where d is a negative integer having two distinct representations of the form $a^2 - 4b^3$ with integers a and b subject to some additional constraints. This is the approach Byeon has taken, for which he used some technical results due to Yu [20]. Craig also showed how to create infinitely many real quadratic fields of 3-rank at least two, but his examples here are of the form $\mathbb{Q}(\sqrt{g(a)})$, where $g(X)$ is a certain polynomial with integer coefficients in one variable of degree 24, and a is an integer. While a lower bound on such count has not yet appeared explicitly in the literature, it is clear that the number of such fields will not be as large as indicated by our Theorem 1.

Our second goal is to prove an extension of a result first established in [8] concerning infinite parametrized families of (both real and complex) quadratic number fields of 3-rank at least 2. We achieve this goal in Section 2. The results from Section 2 are then used in Section 3 together with a result on squarefree values of binary forms which appears in [18] (see also [2,9,10,12,13]) to prove Theorem 1. We point out that for the actual proof of Theorem 1 we do not need the full strength of the results proved in Section 2. The particular result from [8] (which is Theorem 2) suffices for the proof of Theorem 1. However, we think the results from Section 2 are interesting in their own right.

2. Parametrized families of quadratic fields of 3-rank at least 2

The following theorem was proved in [8].

Theorem 2. *Let a and b be integers with $a \equiv b \equiv \pm 1 \pmod 6$. Let*

$$d(a,b) = a\big(b^2 + 18ab + 108a^2\big)\big(4b^3 - 27ab^2 - 486a^2b - 2916a^3\big). \tag{1}$$

Then $\mathbb{K} = \mathbb{Q}(\sqrt{d(a,b)})$ has 3-rank at least 2.

One of the main ingredients in the proof of Theorem 2 above was the following characterization of Kishi and Miyake [14] of quadratic fields with class number divisible by 3.

Theorem 3. *Let $u, v \in \mathbb{Z}$ and put $g(Z) = Z^3 - uvZ - u^2$. If*

(i) *$d = 4uv^3 - 27u^2$ is not a perfect square*;
(ii) *u and v are relatively prime*;
(iii) *$g(Z)$ is irreducible*;
(iv) *one of the following conditions holds*:

$$\begin{array}{rlll} \text{I.} & 3 \nmid v; & & \\ \text{II.} & 3 \mid v, & uv \not\equiv 3 \pmod 9, & u \equiv v \pm 1 \pmod 9; \\ \text{III.} & 3 \mid v, & uv \equiv 3 \pmod 9, & u \equiv v \pm 1 \pmod{27}, \end{array}$$

then $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ has class number divisible by 3. Conversely, every quadratic number field $\mathbb{K}$ with class number divisible by 3 arises in the above way from a suitable choice of integers u and v.

We shall use Theorem 3 above to construct explicit parametric families of both real and complex quadratic fields whose class group has 3-rank at least 2. The present constructions extend the work in [8]. The idea of the construction is that by Hasse's Theorem [11], a quadratic field $\mathbb{K}$ has class group of 3-rank n if and only if there are exactly $(3^n - 1)/2$ cyclic, cubic, unramified extensions of $\mathbb{K}$. Thus, in order to prove that a quadratic field $\mathbb{K}$ has class group of 3-rank at least 2, it suffices to show that $\mathbb{K}$ has two distinct cyclic, cubic, unramified extensions. With this in mind, we give pairs (u, v) and (x, y), each of them satisfying the conditions of Theorem 3, which give rise to distinct cyclic, cubic, unramified extensions of the same quadratic field $\mathbb{K}$.

2.1. Extending Theorem 2

Choose integers a and b such that $(a, b) \equiv (1, 11), (11, 1) \pmod{30}$. Choose positive integers α and β such that $\alpha \equiv 6, 24 \pmod{30}$ and $\beta \equiv 7, 13, 17, 23 \pmod{30}$, $\gcd(\alpha, a - 18b\beta^2) = 1$, $\gcd(a, \beta) = 1$ and $\gcd(a, b(\alpha^2 - \beta^2)) = 1$. Notice that $\alpha^2 - \beta^2$ and 6 are coprime. Set $c = b(\alpha^2 - \beta^2)$, and observe that $c \equiv a \pmod 6$. Now set

$$\begin{aligned} u &= 8b\beta^2\big(a^2 + 18ac + 108c^2\big), \\ v &= a, \\ x &= 8b\alpha^2\big(a^2 + 18ac + 108c^2\big), \\ y &= a + 18c. \end{aligned} \tag{2}$$

Theorem 4. *Let a, b, α, β be integers satisfying the above congruences and coprimality conditions and let u, v, x, y be defined as above. If we put*

$$d = 8b\beta^2\left(a^2 + 18ac + 108c^2\right)\left(4a^3 - 216b\beta^2\left(a^2 + 18ac + 108c^2\right)\right), \tag{3}$$

then $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ has class group of 3-rank at least 2.

2.2. Preliminary results

We start with the following lemma.

Lemma 1. *Each one of the pairs (u, v) and (x, y) defined above satisfies the hypotheses of Theorem 3; that is, each of $\mathbb{Q}(\sqrt{4uv^3 - 27u^2})$ and $\mathbb{Q}(\sqrt{4xy^3 - 27x^2})$ admit cyclic, cubic, unramified extensions.*

Proof. For condition (i) in Theorem 3, one checks easily that both $4uv^3 - 27u^2$ and $4xy^3 - 27x^2$ have the property that the exponent of 2 in their factorization is odd. Hence, none of them can be a perfect square.

We now check condition (ii). First, note that since $a \equiv c \equiv \pm 1 \pmod 6$ and a and c are coprime, we get that $\gcd(a, 6c) = 1$. Since $u \equiv 864bc^2\beta^2 \pmod a$ and a and β are coprime, we get that u and v are also coprime. We now show that x and y are also coprime. Indeed, since

$$\begin{aligned} x &= 8b\alpha^2\left(a^2 + 18ac + 108c^2\right) \\ &= 8ab\alpha^2(a + 18c) + 864bc^2\alpha^2 \\ &\equiv 864bc^2\alpha^2 \pmod y, \end{aligned}$$

we get that any common prime factor p of x and y would divide $6c\alpha$. If $p \mid 6c$, then since $y \equiv a \pmod{6c}$, we then get that $p \mid a$, which is a contradiction since $\gcd(a, 6c) = 1$. On the other hand, if $p \mid \alpha$ and $p \mid y$, then p divides also $a - 18b\beta^2$, contradicting that fact that α and $a - 18b\beta^2$ are coprime. Thus, x and y are also coprime, so condition (ii) in Theorem 3 is satisfied.

For condition (iii), observe that

$$uv \equiv 8a^3b\beta^2 \equiv -2a^4\beta^2 \equiv a^4 \equiv 1 \pmod 3,$$

and

$$u^2 \equiv a^4b^2\beta^4 \equiv a^6\beta^4 \equiv 1 \pmod 3,$$

so

$$g_1(Z) \equiv Z^3 - uvZ - u^2 \equiv Z^3 - Z - 1 \pmod 3,$$

which shows that $g_1(Z)$ is irreducible modulo 3; hence, irreducible as a polynomial with integer coefficients as well. Notice also that

$$c = b\left(\alpha^2 - \beta^2\right) \equiv 1 - 4 \equiv 2 \pmod 5.$$

Furthermore,

$$\begin{aligned} xy &\equiv 3b\alpha^2\bigl(a^2+3ac+3c^2\bigr)(a+3c) \pmod 5 \\ &\equiv 3\bigl(1+3c+3c^2\bigr)(1+3c) \pmod 5 \\ &\equiv 3(4)(2) \equiv 4 \pmod 5, \end{aligned}$$

and

$$\begin{aligned} x^2 &\equiv 4b^2\alpha^4\bigl(a^2+3ac+3c^2\bigr)^2 \pmod 5 \\ &\equiv 4(1+1+2)^2 \pmod 5 \\ &\equiv 4 \pmod 5, \end{aligned}$$

so

$$g_2(Z) = Z^3 - xyZ - x^2 \equiv Z^3 + Z + 1 \pmod 5$$

is an irreducible polynomial modulo 5; hence, an irreducible polynomial with integer coefficients as well.

Finally, condition (iv) is clearly satisfied since neither a (hence, nor y) is divisible by 3.

This completes the proof of Lemma 1. □

The next lemma follows from Theorem 1 in Llorente and Nart [15].

Lemma 2. *For coprime integers u and v, set $g(Z) = Z^3 - uvZ - u^2$. Assume that g is irreducible and let θ be a root of g. Put $\mathbb{K} = \mathbb{Q}(\theta)$.*

(i) *If $uv \equiv 1 \pmod 3$, then 3 is inert in $\mathbb{K}$.*
(ii) *If the exponent of 3 in the factorization of u is $2n$ (here, $n > 0$) and $uv/3^{2n} \equiv 1 \pmod 3$, then 3 splits completely in $\mathbb{K}$.*

2.3. The proof of Theorem 4

Let θ_1, θ_2 be roots of $g_1(Z) = Z^3 - uvZ - u^2$ and $g_2(Z) = Z^3 - xyZ - x^2$, respectively. Let $\mathbb{L}_1$ and $\mathbb{L}_2$ denote the normal closures of $\mathbb{Q}(\theta_1)$ and $\mathbb{Q}(\theta_2)$, respectively. By Lemma 1, the pairs $(u,\ v)$ and $(x,\ y)$ satisfy the hypotheses of Theorem 3. Thus, $\mathbb{L}_1$ and $\mathbb{L}_2$ are unramified, cyclic, cubic extensions of $\mathbb{Q}(\theta_1)$ and $\mathbb{Q}(\theta_2)$, respectively. Note that the cubic fields $\mathbb{Q}(\theta_1)$ and $\mathbb{Q}(\theta_2)$ have discriminants which differ by a square factor, since

$$\begin{aligned} 4xy^3 - 27x^2 &= x\bigl[4(a+18c)^3 - 27(8b)\alpha^2\bigl(a^2+18ac+108c^2\bigr)\bigr] \\ &= x\left[4a^3 - 27(8c)\left(-a^2 - 18ac - 108c^2 + \frac{b\alpha^2}{c}\bigl(a^2+18ac+108c^2\bigr)\right)\right] \\ &= x\bigl[4a^3 - 27(8b)\bigl(a^2+18ac+108c^2\bigr)\bigl(\beta^2 - \alpha^2 + \alpha^2\bigr)\bigr] \end{aligned}$$

$$\begin{aligned}&= x\big[4v^3 - 27u\big]\\&= \frac{\alpha^2}{\beta^2}u\big(4v^3 - 27u\big).\end{aligned}$$

Thus, $\mathbb{L}_1$ and $\mathbb{L}_2$ are both S_3-extensions of $\mathbb{Q}$ with the same quadratic subfield $\mathbb{Q}(\sqrt{d}\,)$, where d is shown in (3).

We now claim that $\mathbb{L}_1$ and $\mathbb{L}_2$ are not the same. To prove this claim, we show that the prime 3 splits differently in the two fields. For a nonzero integer k, we let $v_3(k)$ be the exponent to which the prime 3 appears in the prime factorization of k. Since $3 \mid \alpha$ and $3 \nmid ab$, we see that $v_3(x) = v_3(8b\alpha^2(a^2 + 18ac + 108c^2)) = 2v_3(\alpha)$. Furthermore,

$$\begin{aligned}\frac{xy}{3^{2v_3(\alpha)}} &= \frac{\alpha^2}{3^{2v_3(\alpha)}}\big[8b\big(a^2 + 18ac + 108c^2\big)(a + 18c)\big]\\&\equiv \big(2a^3b\big)\left(\frac{\alpha}{3^{v_3(\alpha)}}\right)^2 \pmod 3\\&\equiv -2a^4 \equiv 1 \pmod 3.\end{aligned}$$

By Lemma 2, the prime 3 splits completely in $\mathbb{Q}(\theta_2)$, so 3 must also split completely in its normal closure $\mathbb{L}_2$. Now

$$\begin{aligned}uv &= 8ab\beta^2\big(a^2 + 18ac + 108c^2\big)\\&\equiv 2a^3b\beta^2 \pmod 3\\&\equiv a^4 \equiv 1 \pmod 3,\end{aligned}$$

so Lemma 2 implies that 3 is inert in $\mathbb{Q}(\theta_1)$. Thus, 3 does not split completely in $\mathbb{L}_1$, so $\mathbb{L}_1$ and $\mathbb{L}_2$ are not the same. Thus, $\mathbb{K} = \mathbb{Q}(\sqrt{d}\,)$ has two distinct cubic, cyclic, unramified extensions, and therefore its class group has 3-rank at least 2.

Remark. There are several more congruence classes for the parameters a, b, α, β for which $\mathbb{K} = \mathbb{Q}(\sqrt{d}\,)$ can be shown to have 3-rank at least 2 by the same arguments. Here, we list only two other instances. The proofs are left for the reader.

(i) Choose $a \equiv 1, 11 \pmod{30}$, then α and β such that $6 \mid \alpha$, α is coprime to 5, $\beta \equiv \pm 1 \pmod 6$, $\beta \equiv \pm\alpha \pmod 5$ and a and β are coprime. Choose an integer $b \equiv -a \pmod 6$ and $b \equiv \alpha^2 \pmod 5$ and assume that α and $a - 18b\beta^2$ are coprime, and that a and c are also coprime.

(ii) Choose $(a, b) \equiv (1, 17), (11, 7) \pmod{30}$, then $\alpha \equiv 6, 24 \pmod{30}$ and $\beta \equiv 7, 13, 17, 23 \pmod{30}$ such that a and β are coprime, α and $a - 18b\beta^2$ are coprime, and a and c are also coprime.

3. The proof of Theorem 1

The proof of Theorem 1 is based on Theorem 2. More precisely, we count the number of distinct imaginary quadratic fields of discriminant $< x$ in absolute value of the form given in the statement of Theorem 2. For this, we need a result on the number of distinct square-free values of binary forms whose irreducible factors have degree at most 3. There are several such results in the literature. We choose to pick the following one due to Stewart and Topp [18].

3.1. Square-free values of binary forms

The following result appears as Theorem 2 in [2].

Theorem 5. *Let A, B, M be integers with $\gcd(A, M) = \gcd(B, M) = 1$. Let $F(X, Y)$ be a binary form with integer coefficients, nonzero discriminant and degree $r \geqslant 3$. Assume that its irreducible factors are of degree at most 6 and that there is no prime p such that p^2 divides $F(a, b)$ for all integers a and b with $a \equiv A \pmod M$ and $b \equiv B \pmod M$. Let $c_1 < c_2$ and $c_3 < c_4$ be positive numbers. Then there exists a constant c depending on F, M and c_1, c_2, c_3, c_4 such that if $x > x_0$, then the number of distinct squarefree integers of the form $F(a, b)$ as $a \in (c_1x, c_2x)$ and $b \in (c_3x, c_4x)$ are such that $a \equiv A \pmod M$ and $b \equiv B \pmod M$ exceeds cx^2.*

Proof. The above Theorem 5 is a particular case of Theorem 1 in [18], except that the numbers a and b are allowed to vary both in the full interval $(1, x)$ instead of only the dyadic type intervals (c_1x, c_2x) and (c_3x, c_4x). Minor typographical changes lead to the conclusion that the statement remains valid when a and b are in these somewhat restricted ranges also. We do not give details. □

3.2. The proof of Theorem 1

We give all the details of the proof only for the case of the real quadratic fields. At the end, we shall sketch the case of the complex quadratic fields.

Let x be a large positive real number. Let $X = Y = \frac{x^{1/6}}{576}$ and put $\mathcal{I} = (X, 2X)$. Let $\mathcal{A}$ be the set of all pairs $(\alpha, \beta) \in \mathcal{I}^2$ such that if we write $a = 1 + 6\alpha$, $b = 1 + 288\beta$, then the number $d(a, b)$ is square-free, where $d(a, b)$ is the binary form of degree 6 given by (1). It is easy to check that if we put

$$F(x, y) = d(1 + 288x, 1 + 288y),$$

then indeed there is no prime p such that $p^2 \mid F(a, b)$ for all integers a and b. Taking $\alpha = 48\alpha_0$, it follows that we may take $A = B = 1$, $M = 288$, $\alpha_0 \in (X/48, X/24)$ and $\beta \in (X, 2X)$ in Theorem 5, and get that there are

$$> \kappa X^2 > k_1 x^{1/3} \tag{4}$$

distinct squarefree values of the form $d(a, b)$ for such choices of a and b, where κ and κ_1 are some positive constants.

Next, we note that if $(\alpha, \beta) \in \mathcal{I}^2$, then $d(a, b) > 0$. Indeed, with $t = b/a$, this inequality is equivalent to

$$4t^3 > 27t^2 + 486t + 2916.$$

One checks easily that the above inequality is true for all $t > 20$, and since

$$t = \frac{b}{a} > \frac{288\beta}{6\alpha + 1} > \frac{x^{1/6}/2}{x^{1/6}/48 + 1} > 20$$

for large x, we get that indeed $d(a, b) > 0$. Furthermore, again since $20a < b$, we get that

$$d(a, b) < a\bigl(b^2 + (18a)b + \bigl(108a^2\bigr)\bigr)\bigl(4b^3\bigr) < (12a)b^5 < b^6 < x.$$

Hence, for each pair $(\alpha, \beta) \in \mathcal{I}^2$ we have created a real quadratic field $\mathbb{K}_{\alpha,\beta} = \mathbb{Q}(\sqrt{d(1 + 6\alpha, 1 + 288\beta)})$ with $d = d(1 + 6\alpha, 1 + 288\beta) < x$ and the number of such distinct fields is bounded above as shown in (4).

We now sketch the proof for imaginary quadratic fields. For this case, we take $X = Y = \frac{x^{1/6}}{8 \cdot 10^6}$, $\mathcal{I} = (X, 2X)$, and $(a, b) = (1 + 6\alpha, 1 + 6\beta)$, where α, β are both integers in $\mathcal{I}$. One checks that since $b < 2a$ for large a, we have that $d(a, b) < 0$. Furthermore, one checks easily that $|d(a, b)| < x$ for all $(\alpha, \beta) \in \mathcal{I}^2$. Using again Theorem 5, we get that a positive proportion of all the pairs (α, β) lead to distinct square-free values of $d(a, b)$, and obtain the result for imaginary quadratic fields.

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